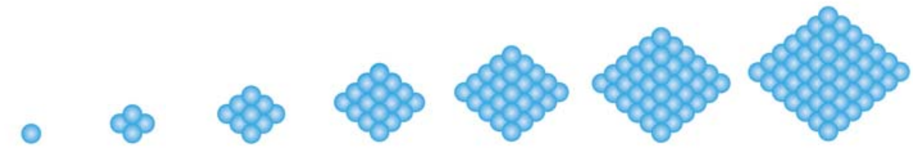
EXAMPLE 3 Model with finite differences

The first seven triangular pyramidal numbers are shown below. Find a polynomial function that gives the *n*th triangular pyramidal number.



$f(1) = 1$ $f(2) = 4$ $f(3) = 10$ $f(4) = 20$ $f(5) = 35$ $f(6) = 56$ $f(7) = 84$

Solution

Begin by finding the finite differences.

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$	
1	4	10	20	35	56	84	**Write function values for equally-spaced *n*-values.**
3	6	10	15	21	28		**First-order differences**
3	4	5	6	7			**Second-order differences**
1	1	1	1				**Third-order differences**

Because the third-order differences are constant, you know that the numbers can be represented by a cubic function of the form $f(n) = an^3 + bn^2 + cn + d$.

By substituting the first four triangular pyramidal numbers into the function, you obtain a system of four linear equations in four variables.

$$a(1)^3 + b(1)^2 + c(1) + d = 1 \quad\Rightarrow\quad a + b + c + d = 1$$
$$a(2)^3 + b(2)^2 + c(2) + d = 4 \quad\Rightarrow\quad 8a + 4b + 2c + d = 4$$
$$a(3)^3 + b(3)^2 + c(3) + d = 10 \quad\Rightarrow\quad 27a + 9b + 3c + d = 10$$
$$a(4)^3 + b(4)^2 + c(4) + d = 20 \quad\Rightarrow\quad 64a + 16b + 4c + d = 20$$

REVIEW SYSTEMS
For help with using matrices to solve linear systems, see p. 210.

Write the linear system as a matrix equation $AX = B$. Enter the matrices A and B into a graphing calculator, and then calculate the solution $X = A^{-1}B$.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 10 \\ 20 \end{bmatrix}}_{B}$$

Calculate $X = A^{-1}B$.

▶ The solution is $a = \frac{1}{6}$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, and $d = 0$. So, the nth triangular pyramidal number is given by $f(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$.

✓ GUIDED PRACTICE for Example 3

4. Use finite differences to find a polynomial function that fits the data in the table.

x	1	2	3	4	5	6
$f(x)$	6	15	22	21	6	−29