EXAMPLE 3 Use zeros to write a polynomial function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and 3 and $2 + \sqrt{5}$ as zeros.

Solution

Because the coefficients are rational and $2 + \sqrt{5}$ is a zero, $2 - \sqrt{5}$ must also be a zero by the irrational conjugates theorem. Use the three zeros and the factor theorem to write $f(x)$ as a product of three factors.

$$f(x) = (x - 3)\left[x - \left(2 + \sqrt{5}\right)\right]\left[x - \left(2 - \sqrt{5}\right)\right] \quad \text{Write } f(x) \text{ in factored form.}$$

$$= (x - 3)\left[(x - 2) - \sqrt{5}\right]\left[(x - 2) + \sqrt{5}\right] \quad \text{Regroup terms.}$$

$$= (x - 3)[(x - 2)^2 - 5] \quad \text{Multiply.}$$

$$= (x - 3)[(x^2 - 4x + 4) - 5] \quad \text{Expand binomial.}$$

$$= (x - 3)(x^2 - 4x - 1) \quad \text{Simplify.}$$

$$= x^3 - 4x^2 - x - 3x^2 + 12x + 3 \quad \text{Multiply.}$$

$$= x^3 - 7x^2 + 11x + 3 \quad \text{Combine like terms.}$$

CHECK You can check this result by evaluating $f(x)$ at each of its three zeros.

$$f(3) = 3^3 - 7(3)^2 + 11(3) + 3 = 27 - 63 + 33 + 3 = 0 \checkmark$$

$$f\left(2 + \sqrt{5}\right) = \left(2 + \sqrt{5}\right)^3 - 7\left(2 + \sqrt{5}\right)^2 + 11\left(2 + \sqrt{5}\right) + 3$$

$$= 38 + 17\sqrt{5} - 63 - 28\sqrt{5} + 22 + 11\sqrt{5} + 3$$

$$= 0 \checkmark$$

Since $f\left(2 + \sqrt{5}\right) = 0$, by the irrational conjugates theorem $f\left(2 - \sqrt{5}\right) = 0$. ✓

✓ GUIDED PRACTICE for Example 3

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

5. $-1, 2, 4$ **6.** $4, 1 + \sqrt{5}$ **7.** $2, 2i, 4 - \sqrt{6}$ **8.** $3, 3 - i$

DESCARTES' RULE OF SIGNS French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

KEY CONCEPT *For Your Notebook*

Descartes' Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.